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The Black Hole Entropy Can Be Smaller than $A/4$

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Abstract

The coupling of a Nambu-Goto string to gravity allows for Schwarzschild black holes whose entropy to area relation is $S = (A/4)(1-4\mu)$, where μ is the string tension.

It is well known that to compute thermal correlation functions and partition functions in field theory in flat Minkowski spacetime one can use path integrals in periodic imaginary time. The period β is the inverse temperature and can be chosen freely.

This method was generalized to compute matter correlation functions in static curved backgrounds. For the Schwarzschild black hole, the analytic continuation to imaginary time defines a Euclidean background everywhere except at the analytic continuation of the horizon, namely the 2-sphere at $r = 2M$, M being the black hole mass. This ambiguity is usually removed by

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assuming that the Euclidean manifold is regular at $r = 2M$. The Euclidean black hole obtained in this way has a periodicity β uniquely defined in terms of the black hole mass. The relation is

$$\beta = 8\pi M. \tag{1}$$

This value coincides with the temperature of the thermal quantum radiation computed for the incipient black hole in the absence of back-reaction [1].

In pioneering work, Gibbons and Hawking [2] extended the analytic continuation to the gravitational action, restricting the hitherto ill-defined path integral over metrics to a saddle point in the Euclidean section. To constitute such a saddle the Euclidean black hole must be regular given that a singularity at $r = 2M$ would invalidate the solution of the Euclidean Einstein equations. The partition function evaluated on this saddle is interpreted as $e^{-\beta F}$, where F is the free energy of the background black hole spacetime. It yields the Bekenstein-Hawking [3] area entropy S for the black hole namely

$$S = \frac{A}{4} \tag{2}$$

where A is the area of the event horizon. In what follows, we shall assume the general validity of the Gibbons-Hawking Euclidean saddle condition. Its significance will be further discussed elsewhere [4].

The relation Eq. (1) between the temperature and the black hole mass is affected by classical surrounding matter but the entropy remains unchanged and is still given by Eq. (2). This value of the entropy seems therefore to depend only on the black hole mass. In this letter it will be shown that it does not. A different relation between entropy and area will be presented when a conical singularity is present in the Euclidean section at $r = 2M$.

Many authors [5, 6, 7, 8, 9, 10] have introduced a conical singularity at $r = 2M$. This modifies the Euclidean periodicity of the black hole and therefore the temperature. However if the source producing this singularity is not taken into account, the Euclidean black hole solution is not a saddle point of the functional integral. A true Euclidean saddle point can nonetheless be maintained by introducing an elementary string in the action. The conical singularity arises from a string "instanton" and the associated deficit angle is determined by the string tension. The temperature depends on the string tension and it now necessarily entails a variation of the entropy versus area

ratio. This ratio takes value in the interval $[0, 1/4]$, the lower limit being approached when the cone degenerates. At fixed string tension, the relation between entropy and area remains insensitive to the introduction of classical surrounding matter.

The Lorentzian action for gravity coupled to matter fields is taken to be

$$I = \frac{1}{16\pi}\int_M \sqrt{|g|}R - \frac{1}{8\pi}\int_{\partial M} \sqrt{|h|}K + I_{matter}. \tag{3}$$

Here $\frac{1}{16\pi}\int_M \sqrt{|g|}R$ is the usual Einstein-Hilbert action, K is the trace of the extrinsic curvature on the boundary ∂M of the four dimensional manifold M, and h the determinant of the induced metric.

The introduction of the K-term requires explanation. We will justify it briefly and refer the interested reader to the recent detailed discussion of Hawking and Horowitz [11]. The Einstein-Hilbert action contains second-order derivatives of the metric. If the system evolves between two non intersecting spacelike hypersurfaces these second derivative terms can be transformed by partial integration into boundary terms on these spacelike surfaces and on timelike surfaces. Explicitly these boundary terms stem from the integral of the four-divergence $\partial_\mu \omega^\mu$ where

$$\omega^\mu = -\frac{1}{16\pi}\left(\partial_\nu(\sqrt{|g|}g^{\mu\nu}) + g^{\mu\nu}\partial_\nu\sqrt{|g|}\right). \tag{4}$$

Their contribution to the action Eq. (3) is cancelled by the K-term. The absence of boundary terms on the spacelike surfaces is necessary for the consistency of the Hamiltonian formalism. However, for the asymptotically flat spaces considered here, the K-term introduces divergences at spacelike infinity. These can be removed by subtracting from Eq. (3) a K-term at infinity in flat space. It can then be verified that the subtracted action

$$\begin{aligned} I - I_0 &= \frac{1}{16\pi}\int_M \sqrt{|g|}R - \frac{1}{8\pi}\int_{\partial M} \sqrt{|h|}K \\ &\quad + \frac{1}{8\pi}\int_{(\partial M)_\infty} \sqrt{|h_0|}K_0 + I_{matter} \end{aligned} \tag{5}$$

yields the correct ADM mass as the on-shell value of the Hamiltonian. The action Eq. (5) can now be written as a Hamiltonian action and the path integral over metrics can be formally defined.

We will consider a system where the matter is an elementary Nambu-Goto string. Its action is given by

$$I_{matter} \equiv I_{string} = -\mu \int d^2\sigma \sqrt{|\gamma|}, \tag{6}$$

where μ is the string tension and γ is the determinant of the induced metric on the worldsheet:

$$\gamma_{ab}(z) = g_{\mu\nu}(z)\partial_a z^\mu \partial_b z^\nu. \tag{7}$$

In the presence of a string, the Lorentzian Einstein equations still admit ordinary black hole solutions corresponding to trivial zero string area. The continuation of these solutions to imaginary time is given by the metric

$$ds^2 = \left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\Omega^2 \tag{8}$$

for $r > 2M$. To determine the Euclidean background at $r = 2M$ we impose the Euclidean Einstein equations. As previously discussed, the free energy will then be computed on the saddle of the Euclidean action.

The action Eq. (3) with matter term Eq. (6) can be extended to Euclidean metrics. The subtracted Euclidean action reads

$$\begin{aligned} \widetilde{I} - \widetilde{I}_0 &= -\frac{1}{16\pi}\int_{\widetilde{M}} \sqrt{g}R + \frac{1}{8\pi}\int_{\partial\widetilde{M}} \sqrt{h}K \\ &\quad -\frac{1}{8\pi}\int_{(\partial\widetilde{M})_\infty} \sqrt{h_0}K_0 + \mu\int d^2\sigma\sqrt{\gamma}. \end{aligned} \tag{9}$$

Here the world sheet has the topology of a 2-sphere. Because the Euclidean black holes Eq. (8) have only one boundary, namely at infinity, we must take $\partial\widetilde{M} = (\partial\widetilde{M})_\infty$ in the K-term. The K_0-term subtraction has to be performed with the extrinsic curvature in flat Euclidean space. The independent variables in Eq. (9) are the components of the metric $g_{\mu\nu}$ and the string coordinates z^μ. The variation of the action with respect to $g_{\mu\nu}$ gives the Euclidean Einstein equations:

$$R_{\mu\nu}(x) - \frac{1}{2}g_{\mu\nu}(x)\; R(x) = 8\pi T_{\mu\nu}(x) \tag{10}$$

where

$$T^{\mu\nu}(x) = -\mu\int d^2\sigma\sqrt{\gamma}\gamma^{ab}\partial_a z^\mu\partial_b z^\nu \frac{1}{\sqrt{g(z)}}\delta^4(x-z). \tag{11}$$

Variations with respect to z^μ give rise to the stationary area condition for the string.

The Einstein equations Eq. (10) still admit ordinary Euclidean black hole solutions corresponding to zero string area. The Euclidean space is regular at $r = 2M$ and the t-periodicity is

$$\beta_H = 8\pi M. \tag{12}$$

However there exists a non-trivial solution to the string equations of motion when the string wraps around the Euclidean continuation of the horizon, a sphere at $r = 2M$. All solutions are correctly described by the metric Eq. (8) but the non-trivial one has a curvature singularity at $r = 2M$.

The trace of Einstein equations Eq. (10) gives

$$\int_{\widetilde{M}} \sqrt{g} R = 16\pi\mu A, \tag{13}$$

where A, the area of the string, is equal to the area of the horizon, a two sphere at $r = 2M$. The entire contribution to the integral comes from the singularity at $r = 2M$. To evaluate $\int \sqrt{g} R$ when R is zero everywhere except at $r = 2M$, one can consider an infinitesimal tubular neighborhood $S^2 \times D$ of $r = 2M$ [7, 8]. This gives

$$\int_{S^2 \times D} \sqrt{g}\ R = A \int_D \sqrt{{}^{(2)}g}\ {}^{(2)}R. \tag{14}$$

From Eq. (13) we have

$$\frac{1}{4\pi} \int_D \sqrt{{}^{(2)}g}\,{}^{(2)}R = 4\mu. \tag{15}$$

This result and the Gauss-Bonnet theorem for disc topology tell us that there is a conical singularity with deficit angle $2\pi\eta$ such that

$$\eta = 4\mu. \tag{16}$$

This deficit angle is the sole effect of the string instanton. The periodicity in t is now:

$$\beta = \beta_H\ (1 - 4\mu). \tag{17}$$

The string instanton has raised the global temperature from β_H^{-1} to β^{-1}.

We now evaluate the free energy of the black hole. The contribution of the string term to the action Eq. (9) exactly cancels the contribution of the Einstein term as seen from Eq. (13). The boundary terms at asymptotically large $r = r_\infty$ are thus the only ones contributing to βF. Using Eqs. (4) and (8) we find

$$\frac{1}{8\pi}\int_{\partial\widetilde{M}=(\partial\widetilde{M})_\infty}\sqrt{h}K = -\beta\left(r_\infty\left(1-\frac{2M}{r_\infty}\right)+\frac{M}{2}\right). \tag{18}$$

The subtracted term is computed similarly in the flat metric

$$ds^2 = \left(1-\frac{2M}{r_\infty}\right)dt^2 + dr^2 + r^2 d\Omega^2, \tag{19}$$

where t has the periodicity β given by Eq. (17). The subtraction term is

$$\frac{1}{8\pi}\int_{(\partial\widetilde{M})_\infty}\sqrt{h_0}K_0 = -\beta r_\infty\left(1-\frac{2M}{r_\infty}\right)^{\frac{1}{2}}. \tag{20}$$

The free energy is given by

$$F = \beta^{-1}(\widetilde{I}-\widetilde{I}_0)_{saddle} = \frac{M}{2}. \tag{21}$$

Note that the free energy has the same value it had in the absence of the string instanton. However, using Eq. (17), the entropy

$$S = \beta^2\frac{dF}{d\beta} = \frac{\beta^2}{2}\frac{dM}{d\beta} \tag{22}$$

is now given by

$$S = (1-4\mu)\frac{A}{4}. \tag{23}$$

This is our central result. The introduction of the string has enabled us to define a black hole with fixed mass M at a temperature other than the usual β_H^{-1}. When the temperature is not the Hawking temperature, the entropy changes from $A/4$ to Eq. (23).

It follows from Eq. (17) and Eq. (23) that the product of the entropy and the temperature is constant for a given mass, independent of the string tension.

$$\beta^{-1}S = \beta_H^{-1}S_H = \frac{M}{2}, \tag{24}$$

where $S_H = A/4$.

To complete the thermodynamic analysis, we verify using Eqs.(21) and (24) that the energy of the solution is unchanged by the presence of the string instanton.

$$E = F + \beta^{-1} S = M. \tag{25}$$

It can be shown [4] that the entropy Eq. (23) is not affected by the presence of classical matter surrounding the black hole. In this case, Eq. (24) becomes

$$\beta^{-1} S = \beta_{Hm}^{-1} S_H, \tag{26}$$

where β_{Hm}^{-1} is the inverse Hawking temperature in presence of matter. This indicates that the entropy is a genuine property of the horizon.

We now comment about the interpretation of our result from the Lorentzian viewpoint. As stated in the introductory paragraphs of this Letter, there is an ambiguity in making the analytic continuation to imaginary time at the horizon of the Lorentzian black hole metric. This ambiguity was removed by requiring that the Euclidean metric be a saddle point of the Euclidean action. This requirement imposes that one considers all available saddles. The solution with the string instanton is one such saddle[5] . It corresponds to the original Lorentzian black hole in a different thermodynamic state than the solution with no wrapping. This new state is characterized by a different temperature than the Hawking temperature and therefore by different boundary conditions for quantum fields in the Lorentzian background. It is well known that the Hawking temperature corresponds to the regularity of the expectation value of the energy momentum tensor of quantum fields in the Lorentzian background [12]. Therefore the new boundary conditions, corresponding to the temperature determined by the string instanton, lead to well defined singularities in the energy momentum tensor of quantum fields on the horizon. This will be discussed in more detail in a separate publication [4]. These singularities are quantum effects in a usual Schwarzschild background which is regular on the horizon. Of course, these considerations do not take into account the backreaction.

In this Letter, we have shown that the area entropy of an eternal Schwarzschild black hole in the presence of a string instanton differs from the usual

[5] Multi-instantons could be considered. They have different temperature and correspond to distinct thermodynamic states. Note however that the deficit angle cannot exceeds its maximal value of 2π for which the cone degenerates.

value $A/4$ and depends on the value of the string tension. Such a black hole differs from an ordinary Schwarzschild black holes only by the instanton effect. One cannot distinguish between them through their mass or even through their Lorentzian metrics. They differ through quantum effects, not in classical quantities such as $M/2$, the product of β^{-1} and S. Indeed the temperature is proportional to $\hbar$ while the entropy is inversely proportional to $\hbar$. In this sense the instanton provides a quantum hair [5] which affects the expectation values of operators. The string selects from all possible black holes of mass M a subset distinguishable by quantum effects and characterized by a smaller entropy. This fact give credence to the interpretation of the area entropy as a counting of states and points towards a possible retrieval by quantum effects of the information concealed by, or stored in, the black hole horizon.

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References

[1] S.W. Hawking, Commun. Math. Phys. **43** (1975) 199.

[2] G.W. Gibbons and S.W. Hawking, Phys. Rev. D **15** (1977) 2752.

[3] J.D. Bekenstein, Phys. Rev. D **7** (1973) 2333.

[4] F. Englert, L. Houart and P. Windey, "Black Hole Entropy and String Instantons," Report No PAR-LPTHE 95-38 and ULB-TH 10/95, hep-th/9507061, 1995.

[5] S. Coleman, J. Preskill and F. Wilczek, Nucl. Phys. **B378** (1992) 175.

[6] F. Dowker, R. Gregory and J. Traschen, Phys. Rev. D **45** (1992) 2762.

[7] M. Bañados, C. Teitelboim and J. Zanelli, Phys. Rev. Lett. **72** (1994) 957.

[8] C. Teitelboim, Phys. Rev. D **51** (1995) 4315.

[9] S. Carlip and C. Teitelboim, Class. Quant. Grav. **12** (1995) 1699.

[10] L. Susskind and J. Uglum, Phys. Rev. D **50** (1994) 2700.

[11] S.W. Hawking and G. Horowitz, "The Gravitational Hamiltonian, Action, Entropy and Surface Terms," Report No DAMTP/R 94-52 and UCSBTH-94-37, gr-qc/9501014, 1995 .

[12] P. Candelas, Phys. Rev. D **21** (1980) 2185.